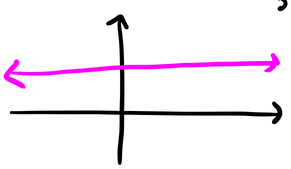
LECTURE: 3-1 DERIVATIVES OF POLYNOMIALS AND EXPONENTIALS

Derivative of a Constant Function: $\frac{d}{dx}(c) =$ 0

Picture: $y=c$ is a horizontal line, as the slope is zero, $\boxed{y'=0}$



def. $y=c$; $f(x)=c$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

Example 1: Find the derivatives of the following functions.

(a) $f(x) = 5.4$

$\boxed{f'(x)=0}$

~~don't do this~~

$f(x)=5.4$
$=0$

↑state it's f'! You just said $5.4=0$!

(b) $g(x) = \pi^7$ some number.

$\boxed{g'(x)=0}$

(c) $h(x) = \ln 2$ also some constant

$\boxed{h'(x)=0}$

Example 2: Using the definition of the derivative, find the derivatives of the following functions.

(a) $f(x) = x^2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x$$

(b) $f(x) = x^3$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2$$

The Power Rule: If n is a positive integer, then $\frac{d}{dx}x^n =$ nx^{n-1}

The power pulls down, exponent decreases by one!

Example 3: Find the derivatives of the following functions.

(a) $f(x) = x^9$

$\boxed{f'(x) = 9x^8}$

(b) $y = x^{99}$

$\boxed{y' = 99x^{98}}$

(c) $\frac{d}{dt}(t^5) = \boxed{5t^4}$

Using the definition of the derivative you can prove that the following derivatives. Does the power rule appear to hold for non-integer exponents as well?

(a) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ $\frac{d}{dx}(x^{-1}) = -1x^{-1-1} = -1x^{-2} = -1/x^2$ ✓

(b) $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$ $\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = 1/2\sqrt{x}$ ✓

Example 4: Differentiate the following functions.

(a) $f(x) = \frac{1}{x^5} = x^{-5}$ (this is still f)

$f'(x) = -5x^{-5-1}$

$f'(x) = -5x^{-6}$

$\boxed{f'(x) = \frac{-5}{x^6}}$

(b) $y = \sqrt[3]{x^5} = (x^5)^{1/3} = x^{5/3}$

$y' = \frac{5}{3}x^{5/3-1}$

$y' = 5/3\, x^{5/3-3/3} = 5/3\, x^{2/3}$

$\boxed{y' = \frac{5}{3}\sqrt[3]{x^2}}$

Using the power rule we can find equations of tangent lines much more quickly! We can also find the **normal line**, which is defined as the line through a point P that is perpendicular to the tangent line at P.

Example 5: Find equations of the tangent line and normal line to the curve $y = x^2\sqrt{x}$ at the point $(1, 1)$.

$y = x^2 \cdot x^{1/2} = x^{4/2}x^{1/2} = x^{5/2}$

$y' = 5/2\, x^{5/2-1} = 5/2\, x^{3/2}$

tan line $m = y'(1) = 5/2 \cdot 1^{3/2} = 5/2$

equation is $y - y_1 = m(x - x_1) \Rightarrow \boxed{y - 1 = \frac{5}{2}(x-1)}$

normal line is $\perp$ to tan line, its m is $-2/5$

line is $\boxed{y - 1 = -2/5\,(x-1)}$

The Constant Multiple Rule: If c is a constant and f is differentiable function then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

$$\lim_{h\to 0} \frac{c\,f(x+h) - c\,f(x)}{h} = c \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= c \cdot f'(x)$$

Example 6: Differentiate the following functions.

(a) $\frac{d}{dx}(5x^7) = 5 \cdot \frac{d}{dx} x^7$

$= 5 \cdot 7x^6$

$= \boxed{35x^6}$

(b) $\frac{d}{dx}(-3\sqrt{x^5}) = -3 \cdot \frac{d}{dx} x^{5/2}$

$= -3 \cdot \frac{5}{2} x^{5/2 - 1}$

$= \boxed{-\frac{15}{2}x^{3/2}} = \boxed{\frac{-15\sqrt{x^3}}{2}}$

The Sum/Difference Rule: If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

$$\frac{d}{dx}[f(x)+g(x)] = \lim_{h\to 0} \frac{f(x+h) + g(x+h) - (f(x)+g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

Example 7: Find the derivative of $y = x^7 + 10x^3 - 7x^2 + 2x - 9$.

$y' = 7x^6 + 10 \cdot 3x^2 - 7 \cdot 2x^1 + 2 \cdot 1x^0 - 0$

$\boxed{y' = 7x^6 + 30x^2 - 14x + 2}$

this means an ordered pair.

means the tangent line has slope zero.

Example 8: Find the points on the curve $y = x^4 - 2x^2 + 4$ where the tangent line is horizontal.

① find $y' = 4x^3 - 4x$

② set $y' = 0$ + solve $\Rightarrow 0 = 4x^3 - 4x$

$0 = 4x(x^2 - 1)$

$0 = 4x(x+1)(x-1) \Rightarrow x = 0, \pm 1$

③ get y-coordinates: $x = -1 \Rightarrow y = 1 - 2 + 4 = 3$ $\boxed{(-1,3)}$

$x = 0 \Rightarrow y = 4$ $\boxed{(0,4)}$

$x = 1 \Rightarrow y = 1 - 2 + 4 = 3$ $\boxed{(1,3)}$

Example 9: Find the derivatives of the following functions.

you need to do some algebra so the power rule applies!

(a) $y = (5x^2 - 2)^2$

$y = (5x^2 - 2)(5x^2 - 2)$

$y = 25x^4 - 20x^2 + 4$

$y' = 25 \cdot 4x^3 - 20 \cdot 2x + 0$

$\boxed{y' = 100x^3 - 40x}$

(b) $f(x) = \dfrac{\sqrt{x} + 2x - 3}{x^3} = (x^{1/2} + 2x^1 - 3)x^{-3}$

$= x^{1/2}x^{-3} + 2x\,x^{-3} - 3x^{-3}$ those exponents add!

$= x^{-5/2} + 2x^{-2} - 3x^{-3}$

$f'(x) = -\frac{5}{2}x^{-5/2-1} + 2(-2)x^{-2-1} - 3(-3)x^{-3-1}$

$\boxed{f'(x) = -\frac{5}{2}x^{-7/2} - 4x^{-3} + 9x^{-4}}$

Derivative of the Natural Exponential Function: $\dfrac{d}{dx}e^x = e^x$

Example 10: Find the derivatives of the following functions.

(a) $f(t) = \sqrt{3t} + \sqrt{\dfrac{3}{t}}$

you can't (yet) do derivatives if there's a number inside... so fix it!

$= \sqrt{3}\sqrt{t} + \sqrt{3}/\sqrt{t}$

$= \sqrt{3}\,t^{1/2} + \sqrt{3}\,t^{-1/2}$

$f'(t) = \sqrt{3} \cdot \frac{1}{2}t^{1/2-1} + \sqrt{3}\left(\frac{-1}{2}\right)t^{-1/2-1}$

$\boxed{f'(t) = \frac{\sqrt{3}}{2}t^{-1/2} - \frac{\sqrt{3}}{2}t^{-3/2}}$

(b) $f(x) = e^{x+2} + 4$

$= e^x \cdot e^2 + 4$

$= e^2 e^x + 4$

$f'(x) = e^2 \cdot e^x + 0$

$\boxed{f'(x) = e^{x+2}}$

Example 11: At what point on the curve $y = e^x$ is the tangent line parallel to the line $y - 5x = 2$?

$y - 5x = 2 \Rightarrow y = 5x + 2$ has slope $m = 5$

$y' = e^x$ is slope of tangent line to $y = e^x$,

when does $5 = e^x \Rightarrow x = \ln 5$

point: $x = \ln 5$, $y = e^{\ln 5} = 5$ $\boxed{(\ln 5, 5)}$

Example 13: Biologists have proposed a cubic function to model the length L of an Alaskan rockfish at age A:

$$L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21$$

where L is measured in inches and A in years. Calculate $\frac{dL}{dA}$ at $A = 12$ and interpret your answer.

$$\frac{dL}{dA} = 0.0155(3A^2) - 0.372(2A) + 3.95$$

$$\frac{dL}{dA} = 0.0465A^2 - 0.744A + 3.95$$

$$\left.\frac{dL}{dA}\right|_{A=12} = 0.0465(12^2) - 0.744(12) + 3.95$$

$$= \boxed{1.718 \text{ in/year}}$$

Rockfish growing @ rate of 1.718 in/year.

Example 14: The equation of motion of a particle is $s = 2t^3 - 15t^2 + 36t + 1$. Find the velocity and acceleration functions. Then, determine the acceleration when the velocity is zero.

$$v(t) = s'(t) = \boxed{6t^2 - 30t + 36}$$

$$a(t) = v'(t) = \boxed{12t - 30}$$

$$v(t) = 0 \Rightarrow 0 = 6(t^2 - 5t + 6)$$

$$0 = (t-2)(t-3)$$

$$t = 2, 3$$

$$a(2) = 24 - 30 = \boxed{-6 \text{ m/s}^2}$$

$$a(3) = 36 - 30 = \boxed{6 \text{ m/s}^2}$$

Example 15: Find the following limits.

(a) $\lim_{h \to 0} \frac{(2+h)^5 - 32}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$

$f(x) = x^5$, $a = 2$

This is the derivative of $f(x)$ @ $a=2$.

Thus, the limit is $f'(2) = 5 \cdot 2^4$

$$= 5(16)$$

$$= \boxed{80}$$

(b) $\lim_{x \to 1} \frac{x^{99} - 1}{x - 1} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$f(x) = x^{99}$, $a = 1$

This is the deriv. of $f(x)$ w/ $a=1$ plugged in.

$f'(x) = 99x^{98}$

$f'(1) = \boxed{99}$